

Instructions: Complete and precise solutions will receive full credit. Imprecise or partial solutions will receive very little or no credit. In writing down partial solutions try to indicate the gaps as clearly as possible. If you are using a result from class then please state and verify the hypothesis required for conclusion.

1. **(15 points)** Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i + 1 \\ 1 - p_i & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where we assume $0 < p_i < 1$ for every $i \geq 1$. Let $h : S \rightarrow [0, 1]$ to be given by

$$h(i) = \mathbb{P}(T^0 < \infty \mid X_0 = i),$$

with $T^0 = \inf\{n \geq 0 : X_n = 0\}$. Find h .

2. **(10 points)** Let S_n be the simple symmetric random walk on $\mathbb{Z}$ starting at 0. $T : \Omega \rightarrow \{1, 2, \dots, N\}$ be a stopping time. Show, via a direct proof, that $\mathbb{E}(S_T) = 0$.
3. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ . Assume that $\mu(\{i\}) > 0$ for all $i \in S$.

- (a) **(10 points)** Let $i \in S$ is a recurrent state. Suppose there is a $j \in S$ and $r, s \in \mathbb{N}$ such that $p_{ij}^r > 0$ and $p_{ji}^s > 0$ then show that $j \in S$ is also a recurrent state.
- (b) **(10 points)** For all $i \in S$ assume that $\mu(\{i\}) > 0$ and suppose for all $i, j \in S$ there is a $r, s \in \mathbb{N}$ such that $p_{ij}^r > 0$ and $p_{ji}^s > 0$. Let $T_i = \min\{n \geq 1 : X_n = i\}$. Show that

$$i \in S \text{ is recurrent} \iff \mathbb{P}(T_i < \infty \mid X_0 = j) = 1 \quad \text{for all } j \in S$$

4. **(10 points)** Let X_n be a Markov chain on S with transition matrix P and initial distribution μ . Find the stationary distribution when $S = \{0, 1, 2, \dots, m\}$ and the transition matrix $P = [p_{ij}]_{i,j \in S}$ be given by

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } j = i + 1, i \notin \{0, m\} \\ \frac{1}{2} & \text{if } j = i - 1, i \notin \{0, m\} \\ 1 & \text{if } j = 1, i = 0 \\ 1 & \text{if } j = m - 1, i = m \\ 0 & \text{otherwise} \end{cases}$$